Market models for the smile
Local volatility, local-stochastic volatility

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Outline

- Usable models?

- The local volatility model
- The carry P&L of the LV model
- The delta – the delta of a vanilla option
- Break-even levels for vols of implied vols / covariance of spot and implied volatilities
    - SSR and volatilities of volatilities in the local volatility model

- Local-stochastic volatility models
- A criterion for admissibility
- Examples
- Conclusion

Intro – a practically usable model?

- Imagine we have traded an option of maturity T on an asset S, whose payoff is $f(S_T)$.
- The pricing library supplies a pricing function $P(t, S)$.
- We have no idea of what's been implemented.
- How do we assess whether it's OK to use $P(t, S)$?

- Sanity check 1
  - Set $t = T$; check that $P(t = T, S) \; = \; f(S), \forall S$.

- If OK, then sanity check 2
  - Compute delta: $\Delta \; = \; \frac{dP}{dS}$.
  - P&L of a *short* delta-hegded position during $[t, t + \delta t]$ is:

$$P\&L \; = \; -\Big(P(t + \delta t, S + \delta S) - (1 + r\delta t)P(t, S)\Big) + \Delta\Big(\delta S - (r - q)S\delta t\Big)$$

  - Expand at order 2 in δS, 1 in δt:

$$P\&L \; = \; -\left(-rP + \frac{dP}{dt} + (r - q)S\frac{dP}{dS}\right)\delta t \; - \; \frac{1}{2}\frac{d^2P}{dS^2}\delta S^2$$

Intro – a practically usable model? – 2

- P&L during δt is:

$$P\&L \;=\; -A(t,S)\delta t - B(t,S)\delta S^2$$

  - if $A(t,S) \geq 0,\; B(t,S) \geq 0$ ➩ Always loosing money: no good.
  - if $A(t,S) \leq 0,\; B(t,S) \leq 0$ ➩ Always making money: no good either.

- OK to use $P(t,S)$ only if signs of A and B different, $\forall S, \forall t$.

$$P\&L \;=\; -BS^2\left(\left(\frac{\delta S}{S}\right)^2 + \frac{A}{BS^2}\delta t\right)$$

- Reasonable ansatz, if S is an equity: $\frac{A}{BS^2} \;=\; -cst \;=\; -\widehat{\sigma}^2$. Using expressions of A and B:

$$-rP + \frac{dP}{dt} + (r-q)S\frac{dP}{dS} \;=\; -\widehat{\sigma}^2\frac{1}{2}S^2\frac{d^2P}{dS^2}$$

- This is in fact the BS equation. Carry P&L acquires simple form:

$$P\&L \;=\; -\frac{1}{2}S^2\frac{d^2P}{dS^2}\left(\left(\frac{\delta S}{S}\right)^2 - \widehat{\sigma}^2\delta t\right)$$

Intro – a practically usable model? – 3

- Simple form of $P\&L$ ⇨ simple break-even criterion. Only reason why BS equation used in banks.

- No assumption that equities are lognormal – they are not.
- No assumption that volatility is constant – it is not.
- Not even the assumption of a *process* for S.

- Criterion for breakeven of P&L at order 2 in $\delta S \Rightarrow P$ solves parabolic equation $\Rightarrow$ probabilistic interpretation & P interpreted as an expectation.

- What if there are multiple hedge instruments? Carry P&L reads:

$$P\&L \;=\; -\frac{1}{2}S_iS_j\frac{d^2P}{dS_idS_j}\left(\frac{\delta S_i}{S_i}\frac{\delta S_j}{S_j} \;-\; C_{ij}\delta t\right)$$

  - Criterion for P&L to be nonsensical: C must be positive matrix.
  - There is exist breakeven covariance levels $\forall S, \forall t$ that are *payoff-independent*.
- Important thing: only involves hedge instruments – not model's state variables.
- S_i underliers – or 1 underlying & associated vanilla options.

What's left to do?

- Once option is delta-hedged, we are left with gamma/theta P&L. Total P&L incurred on $[0, T]$:

$$P\&L_T = -\Sigma_i e^{r(T-t_i)} S_i^2 \left.\frac{d^2P}{dS^2}\right|_{t_i,S_i} (r_i^2 - \widehat{\sigma}^2\delta t), \qquad r_i = \frac{\delta S_i}{S_i}$$

- Is this P&L sizeable?

- If S follows a lognormal process with volatility $\widehat{\sigma}$ and $\delta t \to 0$, then $P\&L = 0$.

- Returns of real undelyings (a) do not have exhibit volatility, (b) have non-Gaussian conditional distributions. Set:

$$r_i = \sigma_i Z_i, \qquad E[Z_i^2] = 1$$

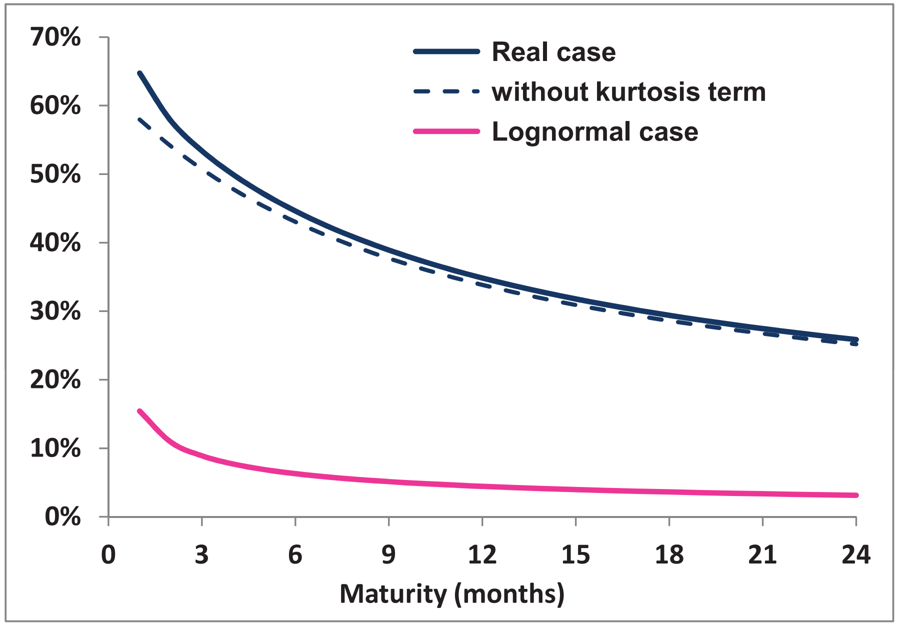
  Then:

$$P\&L_T = -\Sigma_i e^{r(T-t_i)} S_i^2 \left.\frac{d^2P}{dS^2}\right|_{t_i,S_i} (\sigma_i^2 Z_i^2 - \widehat{\sigma}^2\delta t)$$

  - Z_i non-Gaussian ➩ impacts short-maturity options.
  - σ_i random AND correlated ➩ impacts longer-maturity options.

What's left to do? – 2

- Use typical parameters. Stdev($P\&L$) as fraction of price for an ATM option, as a function of maturity:



- For 1y maturity: Black-Scholes: 5%, while $\approx$ 30% in the real case.
- Delta hedging better than nothing – but remaining gamma/theta still too large.

- Gamma needs to be cancelled as well ➪ options are hedged with options.

What's left to do? – conclusion

⇨ P becomes a function of t, S and other derivative prices
  - For example vanilla options: $P(t, S, O_{KT})$.
- This is called "calibration".

- Admissible models are such that the P&L of a delta/vega-hedged option reads:

$$P\&L \;=\; -\frac{1}{2}S_iS_j\frac{d^2P}{dS_idS_j}\left(\frac{\delta S_i}{S_i}\frac{\delta S_j}{S_j} \;-\; C_{ij}\delta t\right)$$

  with C positive (implied) break-even covariance matrix of hedge instruments S_i.
  - C is payoff-independent.
  - Ideally we would like to be able to choose the C_{ij}.

- We call "market models" models satisfying this condition.

- Usually not able to write down SDEs for hedge instruments directly, so condition needs to be checked *a posteriori*.

- 2 examples:
  - Local volatility
  - Local-stochastic volatility

The local volatility model

Local volatility – intro: things heard on the street

- LV model used inconsistently: local vol surface is calibrated today; only to be recalibrated tomorrow.
  ⇨ violates model's assumption of fixed LV surface.
- Trading practice: don't use LV delta – instead compute "sticky-strike" delta: move S, keep implied vols unchanged, recalibrate local vol surface.
  - Rationale: so that vanilla options have BS delta.
- On a scale from dirty to downright ugly, where do we stand?
  - What is the carry P&L of an option position?
  - By the way, what's the delta of a vanilla option?

Local volatility – 1

- Local volatility: simplest model that is able to take as inputs vanilla option prices.
- Provided:
  - no time arbitrage: if zero int. rate: $\frac{dC_{KT}}{dT} \geq 0 \quad \Rightarrow \quad T_1 \leq T_2 \Rightarrow T_1\widehat{\sigma}^2_{KT_1} \leq T_2\widehat{\sigma}^2_{KT_2}$
  - no strike arbitrage: $\frac{d^2C_{KT}}{dK^2} \geq 0$

  there exists a (single) local volatility function $\sigma(t,S)$, given by the Dupire formula:

$$\sigma(t,S)^2 = 2 \left. \frac{\frac{dC}{dT} + qC + (r-q)K\frac{dC}{dK}}{K^2\frac{d^2C}{dK^2}} \right|_{\substack{K=S \\ T=t}}$$

  such that, by using:

$$dS_t = (r-q)S_t dt + \sigma(t,S_t)S_t dW_t$$

  vanilla option prices are recovered.

- Pricing function of LV model reads: $\mathcal{P}(t,S,O_{KT})$ or $P(t,S,\widehat{\sigma}_{KT})$ – no parameter beside time & values of hedge instruments.
- Model assumes fixed $\sigma(t,S)$ while, in practice, local volatility function is recalibrated every day. Does this make any sense?
- What are the deltas (vegas)?

Local volatility – 2

- Pricing equation of the local volatility model reads:

$$\frac{dP^{\text{LV}}}{dt} + (r-q)S\frac{dP^{\text{LV}}}{dS} + \frac{1}{2}\sigma^2(t,S)S^2\frac{d^2P^{\text{LV}}}{dS^2} \;=\; rP^{\text{LV}}$$

  Just like BS equation except $\sigma(t,S)$ instead of cst volatility $\widehat{\sigma}$.

- Solution of PDE is $P^{\text{LV}}(t,S,\sigma)$
- In LV model all instruments have 1-d Markov representation as a function of t,S:

$$\widehat{\sigma}_{KT}(t,S) \;\equiv\; \Sigma^{\text{LV}}_{KT}(t,S,\sigma)$$

- Imagine trading the LV delta:

$$\Delta^{\text{LV}} \;=\; \left.\frac{dP^{\text{LV}}}{dS}\right|_{\sigma}$$

- P&L during δt of delta-hedged option is:

$$P\&L^{\text{LV}} \;=\; -\frac{1}{2}S^2\frac{d^2P^{\text{LV}}}{dS^2}\left(\left(\frac{\delta S}{S}\right)^2 - \sigma^2(t,S)\delta t\right)$$

- $P\&L^{\text{LV}}$ actual P&L only if market implied vols move as prescribed by $\Sigma^{\text{LV}}_{KT}(t,S,\sigma)$.

$$\Rightarrow \quad \Delta^{\text{LV}} \text{ useless}$$

Local volatility – carry P&L

- Let's compute the carry P&L in the LV model.
- Use (black-box) pricing function $P(t, S, \widehat{\sigma}_{KT})$ given by:

$$P\left(t, S, \widehat{\sigma}_{KT}\right) \equiv P^{\mathrm{LV}}(t, S, \sigma\left[t, S, \widehat{\sigma}_{KT}\right])$$

$$P^{\mathrm{LV}}(t, S, \sigma) = P\left(t, S, \Sigma^{\mathrm{LV}}_{KT}(t, S, \sigma)\right)$$

- Start with P&L of *naked* option position:

$$P\&L = -\left[P(t+\delta t,\ S+\delta S,\ \widehat{\sigma}_{KT}+\delta\widehat{\sigma}_{KT}) - (1+r\delta t)P(t, S, \widehat{\sigma}_{KT})\right]$$

- Expand at order 1 in δt, 2 in δS and $\delta\widehat{\sigma}_{KT}$:

$$\begin{aligned} P\&L &= rP\delta t \\ &\quad - \frac{dP}{dt}\delta t - \frac{dP}{dS}\delta S - \frac{dP}{d\widehat{\sigma}_{KT}} \bullet \delta\widehat{\sigma}_{KT} \\ &\quad - \left(\frac{1}{2}\frac{d^2P}{dS^2}\delta S^2 + \frac{d^2P}{dSd\widehat{\sigma}_{KT}} \bullet \delta\widehat{\sigma}_{KT}\delta S + \frac{1}{2}\frac{d^2P}{d\widehat{\sigma}_{KT}d\widehat{\sigma}_{K'T'}} \bullet \delta\widehat{\sigma}_{KT}\delta\widehat{\sigma}_{K'T'}\right) \end{aligned}$$

- Notation $\bullet$ stands for:

$$\frac{df}{d\widehat{\sigma}_{KT}} \bullet \delta\widehat{\sigma}_{KT} \equiv \iint dKdT\ \frac{\delta f}{\delta\widehat{\sigma}_{KT}}\ \delta\widehat{\sigma}_{KT} \equiv \Sigma_{ij}\frac{df}{d\widehat{\sigma}_{K_iT_j}}\ \delta\widehat{\sigma}_{K_iT_j}$$

Local volatility – carry P&L – 2

- $\frac{dP}{dS}, \frac{dP}{dt}$ are computed keeping the $\widehat{\sigma}_{KT}$ fixed – the LV function is *not* fixed.
  - Define sticky-strike delta Δ^{SS}:

$$\Delta^{\mathrm{SS}} = \left.\frac{dP}{dS}\right|_{\widehat{\sigma}_{KT}}$$

- P is not solution of the LV pricing PDE – P^{LV} is:

$$P^{\mathrm{LV}}(t,S,\sigma) = P\left(t,S,\widehat{\sigma}_{KT} = \Sigma^{\mathrm{LV}}_{KT}(t,S,\sigma)\right)$$

- Express derivatives of P^{LV} in terms of derivatives of P:

$$\frac{dP^{\mathrm{LV}}}{dt} = \frac{dP}{dt} + \frac{dP}{d\widehat{\sigma}_{KT}} \bullet \frac{d\Sigma^{\mathrm{LV}}_{KT}}{dt}$$

$$\frac{dP^{\mathrm{LV}}}{dS} = \frac{dP}{dS} + \frac{dP}{d\widehat{\sigma}_{KT}} \bullet \frac{d\Sigma^{\mathrm{LV}}_{KT}}{dS}$$

$$\frac{d^2P^{\mathrm{LV}}}{dS^2} = \left(\frac{d^2P}{dS^2} + 2\frac{d^2P}{dSd\widehat{\sigma}_{KT}} \bullet \frac{d\Sigma^{\mathrm{LV}}_{KT}}{dS} + \frac{d^2P}{d\widehat{\sigma}_{KT}d\widehat{\sigma}_{K'T'}} \bullet \frac{d\Sigma^{\mathrm{LV}}_{KT}}{dS}\frac{d\Sigma^{\mathrm{LV}}_{K'T'}}{dS}\right) + \frac{dP}{d\widehat{\sigma}_{KT}} \bullet \frac{d^2\Sigma^{\mathrm{LV}}_{KT}}{dS^2}$$

- Now insert in LV pricing equation:

$$\frac{dP^{\mathrm{LV}}}{dt} + (r-q)S\frac{dP^{\mathrm{LV}}}{dS} + \frac{1}{2}\sigma^2(t,S)\,S^2\frac{d^2P^{\mathrm{LV}}}{dS^2} = rP^{\mathrm{LV}}$$

  ... to generate relationship involving derivatives of P.

Local volatility – carry P&L – 3

$$\frac{dP}{dt} = rP - (r-q)S\frac{dP}{dS} - \frac{dP}{d\widehat{\sigma}_{KT}} \bullet \mu_{KT}$$
$$- \frac{1}{2}\sigma^2(t,S)\,S^2 \left(\frac{d^2P}{dS^2} + 2\frac{d^2P}{dSd\widehat{\sigma}_{KT}} \bullet \frac{d\Sigma^{\mathrm{LV}}_{KT}}{dS} + \frac{d^2P}{d\widehat{\sigma}_{KT}d\widehat{\sigma}_{K'T'}} \bullet \frac{d\Sigma^{\mathrm{LV}}_{KT}}{dS}\frac{d\Sigma^{\mathrm{LV}}_{K'T'}}{dS} \right)$$

with μ_{KT} given by:

$$\mu_{KT} = \frac{d\Sigma^{\mathrm{LV}}_{KT}}{dt} + \frac{1}{2}\sigma^2(t,S)\,S^2\frac{d^2\Sigma^{\mathrm{LV}}_{KT}}{dS^2} + (r-q)S\frac{d\Sigma^{\mathrm{LV}}_{KT}}{dS}$$

- Now use this expression of $\frac{dP}{dt}$ to rewrite P&L of *naked* option position:

$$P\&L = -\frac{dP}{dS}\left(\delta S - (r-q)S\delta t\right) - \frac{dP}{d\widehat{\sigma}_{KT}} \bullet \left(\delta\widehat{\sigma}_{KT} - \mu_{KT}\delta t\right)$$
$$+ \frac{1}{2}\sigma^2(t,S)\,S^2 \left(\frac{d^2P}{dS^2} + 2\frac{d^2P}{dSd\widehat{\sigma}_{KT}} \bullet \frac{d\Sigma^{\mathrm{LV}}_{KT}}{dS} + \frac{d^2P}{d\widehat{\sigma}_{KT}d\widehat{\sigma}_{K'T'}} \bullet \frac{d\Sigma^{\mathrm{LV}}_{KT}}{dS}\frac{d\Sigma^{\mathrm{LV}}_{K'T'}}{dS} \right)\delta t$$
$$- \left(\frac{1}{2}\frac{d^2P}{dS^2}\delta S^2 + \frac{d^2P}{dSd\widehat{\sigma}_{KT}} \bullet \delta\widehat{\sigma}_{KT}\delta S + \frac{1}{2}\frac{d^2P}{d\widehat{\sigma}_{KT}d\widehat{\sigma}_{K'T'}} \bullet \delta\widehat{\sigma}_{KT}\delta\widehat{\sigma}_{K'T'} \right)$$

Local volatility – carry P&L – 4

- Introduce implied (log-normal) vol of vol of $\widehat{\sigma}_{KT}$:

$$\nu_{KT} = \frac{1}{\Sigma_{KT}^{\mathrm{LV}}} \frac{d\Sigma_{KT}^{\mathrm{LV}}}{dS} S\sigma(t,S)$$

- Rewrite P&L as:

$$\begin{aligned}
P\&L = & -\frac{dP}{dS}\left(\delta S-(r-q)S\delta t\right) - \frac{dP}{d\widehat{\sigma}_{KT}} \bullet \left(\delta\widehat{\sigma}_{KT}-\mu_{KT}\delta t\right) \\
& -\frac{1}{2}S^2\frac{d^2P}{dS^2}\left[\frac{\delta S^2}{S^2}-\sigma^2(t,S)\,\delta t\right] \\
& -\frac{d^2P}{dS d\widehat{\sigma}_{KT}} \bullet S\widehat{\sigma}_{KT}\left[\frac{\delta S}{S}\frac{\delta\widehat{\sigma}_{KT}}{\widehat{\sigma}_{KT}}-\sigma(t,S)\,\nu_{KT}\delta t\right] \\
& -\frac{1}{2}\frac{d^2P}{d\widehat{\sigma}_{KT} d\widehat{\sigma}_{K'T'}} \bullet \widehat{\sigma}_{KT}\widehat{\sigma}_{K'T'}\left[\frac{\delta\widehat{\sigma}_{KT}}{\widehat{\sigma}_{KT}}\frac{\delta\widehat{\sigma}_{K'T'}}{\widehat{\sigma}_{K'T'}}-\nu_{KT}\nu_{K'T'}\delta t\right]
\end{aligned}$$

- Only uses market observables: $P(t,S,\widehat{\sigma}_{KT})$ – no LV function involved.
- P&L expression is that of market model.
  - Variance/covariance breakeven levels are well-defined, payoff-independent, and make up a positive covariance matrix.
  - Delta is sticky-strike delta $\frac{dP}{dS}$, vegas simple vegas.

Local volatility – carry P&L – 5

- $\widehat{\sigma}_{KT} \equiv$ implied vol plays no special role. Use instead price O_{KT}: $\mathcal{P}(t,S,O_{KT})$.

$$P(t,S,\widehat{\sigma}_{KT}) = \mathcal{P}\big(t,S,O_{KT} = P^{\mathrm{BS}}_{KT}(t,S,\widehat{\sigma}_{KT})\big)$$

$$P^{\mathrm{LV}}(t,S,\sigma) = \mathcal{P}\big(t,S,\Omega^{LV}_{KT}(t,S,\sigma)\big)$$

  $\Omega^{LV}_{KT}(t,S,\sigma)$ price in LV model with LV function σ.
  Everything same as before, except $\widehat{\sigma}_{KT} \to O_{KT}$, $\Sigma^{\mathrm{LV}}_{KT} \to \Omega^{\mathrm{LV}}_{KT}$.

- Drift μ_{KT} simplifies:

$$\mu_{KT} = \frac{d\Omega^{\mathrm{LV}}_{KT}}{dt} + \frac{1}{2}\sigma^2(t,S)\,S^2\frac{d^2\Omega^{\mathrm{LV}}_{KT}}{dS^2} + (r-q)S\frac{d\Omega^{\mathrm{LV}}_{KT}}{dS} = r\Omega^{\mathrm{LV}}_{KT} = rO_{KT} \qquad \text{OK}$$

- P&L of naked option position – using only asset prices – no LV function involved:

$$\begin{aligned} P\&L = & -\frac{d\mathcal{P}}{dS}\left(\delta S - (r-q)S\delta t\right) - \frac{d\mathcal{P}}{dO_{KT}} \bullet \left(\delta O_{KT} - rO_{KT}\delta t\right) \\ & -\frac{1}{2}\frac{d^2\mathcal{P}}{dS^2}\left[\delta S^2 - \sigma^2(t,S)\,S^2\delta t\right] \\ & -\frac{d^2\mathcal{P}}{dSdO_{KT}} \bullet \left[\delta S\delta O_{KT} - \sigma^2\,(t,S)\,S^2\frac{d\Omega^{LV}_{KT}}{dS}\delta t\right] \\ & -\frac{1}{2}\frac{d^2\mathcal{P}}{dO_{KT}dO_{K'T'}} \bullet \left[\delta O_{KT}\delta O_{K'T'} - \sigma^2\,(t,S)\,S^2\frac{d\Omega^{LV}_{KT}}{dS}\frac{d\Omega^{LV}_{K'T'}}{dS}\delta t\right] \end{aligned}$$

Local volatility – carry P&L – 6

- Expression of carry P&L – *inclusive of recalibration of local volatility function* – has typical form of market models.
- Hedge instruments all treated on equal footing.
- Implied break-even levels of cross-gammas are payoff-independent – are determined by market smile prevailing at time t.
  - spot/vol correl $= -100\%$
  - vol/vol correl $= 100\%$
  - vol of $\widehat{\sigma}_{KT}$ is $\nu_{KT} = \frac{1}{\Sigma_{KT}^{\text{LV}}} \frac{d\Sigma_{KT}^{\text{LV}}}{dS} S\sigma(t, S)$
- Hedge ratios simply $\left.\frac{d\mathcal{P}}{dS}\right|_{O_{KT}}$ and $\left.\frac{d\mathcal{P}}{dO_{KT}}\right|_{S}$
- Delta of the local volatility model is – market model delta:

$$\Delta^{\text{MM}} = \left.\frac{d\mathcal{P}}{dS}\right|_{O_{KT}}$$

- Delta of vanilla option irrelevant notion.
  - akin to asking model to generate a hedge ratio of one hedging instrument on another hedging instrument.
- Result seems $\approx$ natural; looks like any P that's the solution of a parabolic PDE will do the job – but see pathologies in local/stoch vol models.

Consistency of sticky-strike and market-model deltas

- Use S, O_{KT} ➩ $\mathcal{P}(t, S, O_{KT})$. Hedge ratios $\Delta^{\mathrm{MM}} = \left.\frac{d\mathcal{P}}{dS}\right|_{O_{KT}}, \quad \left.\frac{d\mathcal{P}}{dO_{KT}}\right|_{S}$
- Use $S, \widehat{\sigma}_{KT}$ ➩ $P\left(t, S, \widehat{\sigma}_{KT}\right)$. Hedge ratios $\Delta^{\mathrm{SS}} = \left.\frac{dP}{dS}\right|_{\widehat{\sigma}_{KT}}, \quad \left.\frac{dP}{d\widehat{\sigma}_{KT}}\right|_{S}$
  - $\frac{dP}{d\widehat{\sigma}_{KT}}$ offset by trading *BS-delta-hedged* vanilla options
- Hedge portfolio is:

$$\Pi = \frac{d\mathcal{P}}{dS} S + \frac{d\mathcal{P}}{dO_{KT}} \bullet O_{KT}$$

- Rewrite in terms of delta-hedged vanillas:

$$\Pi = \left[\frac{d\mathcal{P}}{dS} + \frac{d\mathcal{P}}{dO_{KT}} \bullet \frac{dP_{KT}^{\mathrm{BS}}}{dS}\right] S + \frac{d\mathcal{P}}{dO_{KT}} \bullet \left[O_{KT} - \frac{dP_{KT}^{\mathrm{BS}}}{dS} S\right]$$

- Spot hedge ratio?
  - Move spot + move vanilla prices by their Black-Scholes deltas
    akin to: move vanilla prices keeping implied vols fixed ➩ sticky strike delta

$$\Delta^{\mathrm{SS}} = \frac{d\mathcal{P}}{dS} + \frac{d\mathcal{P}}{dO_{KT}} \bullet \frac{dP_{KT}^{\mathrm{BS}}}{dS}$$

- Once hedge portfolio broken down into underlying + *naked* vanilla options, delta always equal to $\Delta^{\mathrm{MM}} = \left.\frac{d\mathcal{P}}{dS}\right|_{O_{KT}}$.
- Nothing fundamental about Δ^{SS} – tied to a particular representation of vanilla option prices.

So, what is the LV model?

- The LV model is a usable model. It is a market model for the underlying and vanilla options
  ... that happens to have a 1-d Markov representation in terms of (t, S).
- This is a mathematical technicality – of which the LV function is a by-product – that facilitates pricing. Nothing fundamental.
- Daily recalibration of LV function is exactly how it has to be used.
- Consequences of 1-d Markov representation:
  - The break-even covariance matrix is of rank 1 – correls $= 100\%$.
  - No control on break-even levels of volatilities of implied volatilities. They are set by the configuration of S, $\widehat{\sigma}_{KT}$ and will vary unpredictably.
  - Like them, use model – don't like them, don't use model.
- LV model completely specified by feeding in the values of the hedge instruments – no parameters whatsoever.
- This is how much we can get in a model with a 1-d Markov representation.

Using the LV model

- What's left before we can use LV model? Output the ν_{KT}, see if we like them.
  - More practical to look at implied vols for floating strike – fixed moneyness.
- Look at vols of vols and spot/vol covariances.
- For ATMF vol $\widehat{\sigma}_{F_T T}$ equivalently look at SSR $\mathcal{R}_T$

$$\mathcal{R}_T = \frac{1}{\mathcal{S}_T} \frac{\left\langle d\widehat{\sigma}_{F_T T}\ d\ln S \right\rangle}{\langle (d\ln S)^2 \rangle} = \frac{1}{\mathcal{S}_T} \frac{d\widehat{\sigma}_{F_T T}}{d\ln S}$$

$$\mathcal{S}_T = \left. \frac{d\widehat{\sigma}_{KT}}{d\ln K} \right|_{F_T}$$

- Vol of vol:

$$\frac{d\widehat{\sigma}_{F_T T}}{\widehat{\sigma}_{F_T T}} = \frac{1}{\widehat{\sigma}_{F_T T}} \frac{d\widehat{\sigma}_{F_T T}}{d\ln S} d\ln S_t = \frac{d\widehat{\sigma}_{F_T T}}{d\ln S} \frac{\sigma(t,S)}{\widehat{\sigma}_{F_T T}} dW_t$$

  Thus:

$$\text{vol}(\widehat{\sigma}_{F_T T}) = \mathcal{R}_T \mathcal{S}_T \left(\frac{\widehat{\sigma}_{F_0 0}}{\widehat{\sigma}_{F_T T}} \right)$$

- Assume following expression for LV function:

$$\sigma(t,S) = \sigma(t) + \alpha(t) x + \frac{\beta(t)}{2} x^2, \quad x = \ln\left(\frac{S}{F_t}\right)$$

  and calculate $\mathcal{S}_T, \mathcal{R}_T$ at order 1 in $\alpha(t), \beta(t)$.

Expansion of Implied volatilities

- Consider an LV model – Model 1: LV function $\sigma_1(t,S)$, pricing function $P_1(t,S)$.

$$\frac{dP_1}{dt} + (r-q)S\frac{dP_1}{dS} + \frac{1}{2}\sigma_1^2(t,S)\,S^2\frac{d^2P_1}{dS^2} = rP_1$$

- Now consider arbitrary diffusive model – Model 2: instantaneous volatility σ_{2t}.

$$dS_t = (r-q)\,S_t dt + \sigma_{2t}S_t dW_t$$

- Consider process Q_t defined by:

$$Q_t = e^{-rt}P_1\,(t, S_t)$$

  - At $t=0$, $Q_{t=0} = P_1(0, S_0)$.
  - At $t=T$, $Q_{t=T} = e^{-rT}P_1(T, S_{2T}) = e^{-rT}f(S_{2T})$, that is the final payoff.

$$\begin{aligned} dQ_t &= e^{-rt}\left[\left(-rP_1 + \frac{dP_1}{dt}\right)dt + \frac{dP_1}{dS}dS_t + \frac{1}{2}\frac{d^2P_1}{dS^2}\left\langle dS_t^2\right\rangle\right] \\ &= e^{-rt}\left[\left(-rP_1 + \frac{dP_1}{dt}\right)dt + \frac{dP_1}{dS}dS_t + \frac{1}{2}S_t^2\frac{d^2P_1}{dS^2}\sigma_{2t}^2 dt\right] \\ &= e^{-rt}\left[\frac{dP_1}{dS}(dS_t - (r-q)S_t dt) + \frac{1}{2}S_t^2\frac{d^2P_1}{dS^2}(\sigma_{2t}^2 - \sigma_1^2(t,S_t))dt\right] \end{aligned}$$

$$E_2[dQ_t|t,S_t] = e^{-rt}\frac{S_t^2}{2}\frac{d^2P_1}{dS^2}\left(\sigma_{2t}^2 - \sigma_1(t,S_t)^2\right)dt$$

Expansion of Implied volatilities – 2

$$\begin{aligned} E_2[Q_T] &= Q_0 + \int_0^T E_2[dQ_t] \\ &= P_1(0, S_0) + E_2\left[\int_0^T e^{-rt} \frac{S_t^2}{2} \frac{d^2 P_1}{dS^2} \left(\sigma_{2t}^2 - \sigma_1(t, S_t)^2\right) dt\right] \end{aligned}$$

- So, price in Model 2 given by:

$$P_2(0, S_0, \bullet) = P_1(0, S_0) + E_2\left[\int_0^T e^{-rt} \frac{S_t^2}{2} \frac{d^2 P_1}{dS^2} \left(\sigma_{2t}^2 - \sigma_1(t, S_t)^2\right) dt\right]$$

where $\bullet$ other state variables of Model 2.

- Price(Model 2) = Price(Model 1) + gamma/theta P&L, incurred by hedging payoff using Model 1 with dynamics of S_t generated by Model 2.
  - ⇨ Efficient numerical algorithm for generating vanilla smiles of stochastic volatility models – see book.

- Imagine Model 1 is BS model with implied vol $= \widehat{\sigma}_{KT}$ ⇨ $P_2(0, S_0, \bullet) = P_{\widehat{\sigma}_{KT}}(0, S_0)$

$$0 = E_2\left[\int_0^T e^{-rt} \frac{S_t^2}{2} \frac{d^2 P_{\widehat{\sigma}_{KT}}}{dS^2} \left(\sigma_{2t}^2 - \widehat{\sigma}_{KT}^2\right) dt\right]$$

Expansion of Implied volatilities – 3

- Thus:

$$\widehat{\sigma}^2_{KT} = \frac{E_2\left[\int_0^T e^{-rt} S_t^2 \frac{d^2 P_{\widehat{\sigma}_{KT}}}{dS^2} \sigma_{2t}^2 dt\right]}{E_2\left[\int_0^T e^{-rt} S_t^2 \frac{d^2 P_{\widehat{\sigma}_{KT}}}{dS^2} dt\right]}$$

- Work with variances $u = \sigma^2$. Set $u_0 = \sigma_0^2$ and $\sigma_{2t}^2 = u_0 + \delta u(t, S)$.
- Expand at order 1 in δu: $\widehat{\sigma}^2_{KT} = \sigma_0^2 + \delta(\widehat{\sigma}^2_{KT})$.

$$\begin{aligned}
\widehat{\sigma}^2_{KT} = \sigma_0^2 + \delta(\widehat{\sigma}^2_{KT}) &= \frac{E_{u_0+\delta u}[\bullet\,(u_0 + \delta u)]}{E_{u_0+\delta u}[\bullet]} = u_0 + \frac{E_{u_0+\delta u}[\bullet\ \delta u]}{E_{u_0+\delta u}[\bullet]} \\
&= u_0 + \frac{E_{u_0}[\bullet\ \delta u]}{E_{u_0}[\bullet]} \\
&= \frac{E_{u_0}[\bullet\,(u_0 + \delta u)]}{E_{u_0}[\bullet]}
\end{aligned}$$

- Thus:

$$\widehat{\sigma}^2_{KT} = \frac{E_{\sigma_0}\left[\int_0^T e^{-rt}\ u\,(t, S)\, S^2 \frac{d^2 P_{\sigma_0}}{dS^2} dt\right]}{E_{\sigma_0}\left[\int_0^T e^{-rt}\ S^2 \frac{d^2 P_{\sigma_0}}{dS^2} dt\right]}$$

- Density and gamma available in closed form in BS model.

Dynamics in LV model – 2

- Calculation can be done with deterministic $u_0(t) = \sigma_0^2(t)$. At order 1 in δu:

$$\widehat{\sigma}^2_{KT} = \frac{1}{T}\int_0^T dt \int_{-\infty}^{+\infty} dy \, \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} \, u\left(t, F_t e^{\frac{\omega_t}{\omega_T} x_K + \frac{\sqrt{(\omega_T - \omega_t)\omega_t}}{\sqrt{\omega_T}} y}\right)$$

  where F_t forward for maturity t, $x_K = \ln(\frac{K}{F_T})$ and $\omega_t = \int_0^t \sigma_0^2(\tau) d\tau$.

- Expanding around a cst $\sigma(t) = \sigma_0$: $u_0 = \sigma_0^2$

$$\widehat{\sigma}_{KT} = \frac{1}{T}\int_0^T dt \int_{-\infty}^{+\infty} dy \, \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} \, \sigma\left(t, F_t e^{\frac{t}{T} x_K + \sigma_0 \sqrt{\frac{(T-t)t}{T}} y}\right)$$

⇨ Implied vol ≈ average of local vol around straight line – in ln S – from S to K.

$$\mathcal{S}_T = \left.\frac{d\widehat{\sigma}_{KT}}{d\ln K}\right|_{K=F_T} = \frac{1}{T}\int_0^T \frac{t}{T}\alpha(t)dt$$ "skew averaging" – see also V. Piterbarg

$$\left.\frac{d^2\widehat{\sigma}_{KT}}{d\ln K^2}\right|_{K=F_T} = \frac{1}{T}\int_0^T \left(\frac{t}{T}\right)^2 \beta(t)dt$$

$$\left.\frac{d\widehat{\sigma}_{KT}}{d\ln S}\right|_{K=F_T} = \frac{1}{T}\int_0^T \left(1 - \frac{t}{T}\right)\alpha(t)dt$$

- Cst α, β: $\left.\frac{d\widehat{\sigma}_{KT}}{d\ln K}\right|_{K=F_T} = \frac{\alpha}{2}$, $\left.\frac{d^2\widehat{\sigma}_{KT}}{d\ln K^2}\right|_{K=F_T} = \frac{\beta}{3}$

Dynamics in LV model – 3

- From 1st equation: $\alpha(t) = \frac{d}{dt}(t\mathcal{S}_t) + \mathcal{S}_t$.

$$\frac{d\widehat{\sigma}_{F_T(S)T}}{d\ln S} = \left(\left.\frac{d\widehat{\sigma}_{KT}}{d\ln K}\right|_{K=F_T} + \left.\frac{d\widehat{\sigma}_{KT}}{d\ln S}\right|_{K=F_T} \right) = \frac{1}{T}\int_0^T \alpha(t)dt = \mathcal{S}_T + \frac{1}{T}\int_0^T \mathcal{S}_t dt$$

- Get expression of SSR: $\mathcal{R}_T = \frac{1}{\mathcal{S}_T}\frac{\langle d\widehat{\sigma}_{F_T T}\ d\ln S\rangle}{\langle (d\ln S)^2\rangle} = \frac{1}{\mathcal{S}_T}\frac{d\widehat{\sigma}_{F_T(S)T}}{d\ln S}$:

$$\mathcal{R}_T = 1 + \frac{1}{T}\int_0^T \frac{\mathcal{S}_t}{\mathcal{S}_T} dt$$

- For typical equity smiles, $|\mathcal{S}_t|$ decreases with t ⇨ $\mathcal{R}_T \geq 2$.
- Limiting behavior
  - Short maturities:

$$\lim_{T\to 0} \mathcal{R}_T = 2$$

    Lognormal vol of short ATMF vol = twice the skew.

  - Long maturities – take $\mathcal{S}_T \propto \frac{1}{T^\gamma}$:

$$\lim_{T\to\infty} \mathcal{R}_T = \frac{2-\gamma}{1-\gamma}$$

- For typical value $\gamma = \frac{1}{2}$, $\lim_{T\to\infty} \mathcal{R}_T = 3$.

Dynamics in LV model – 4

- Check approx of SSR on 2 smiles of Eurostoxx50

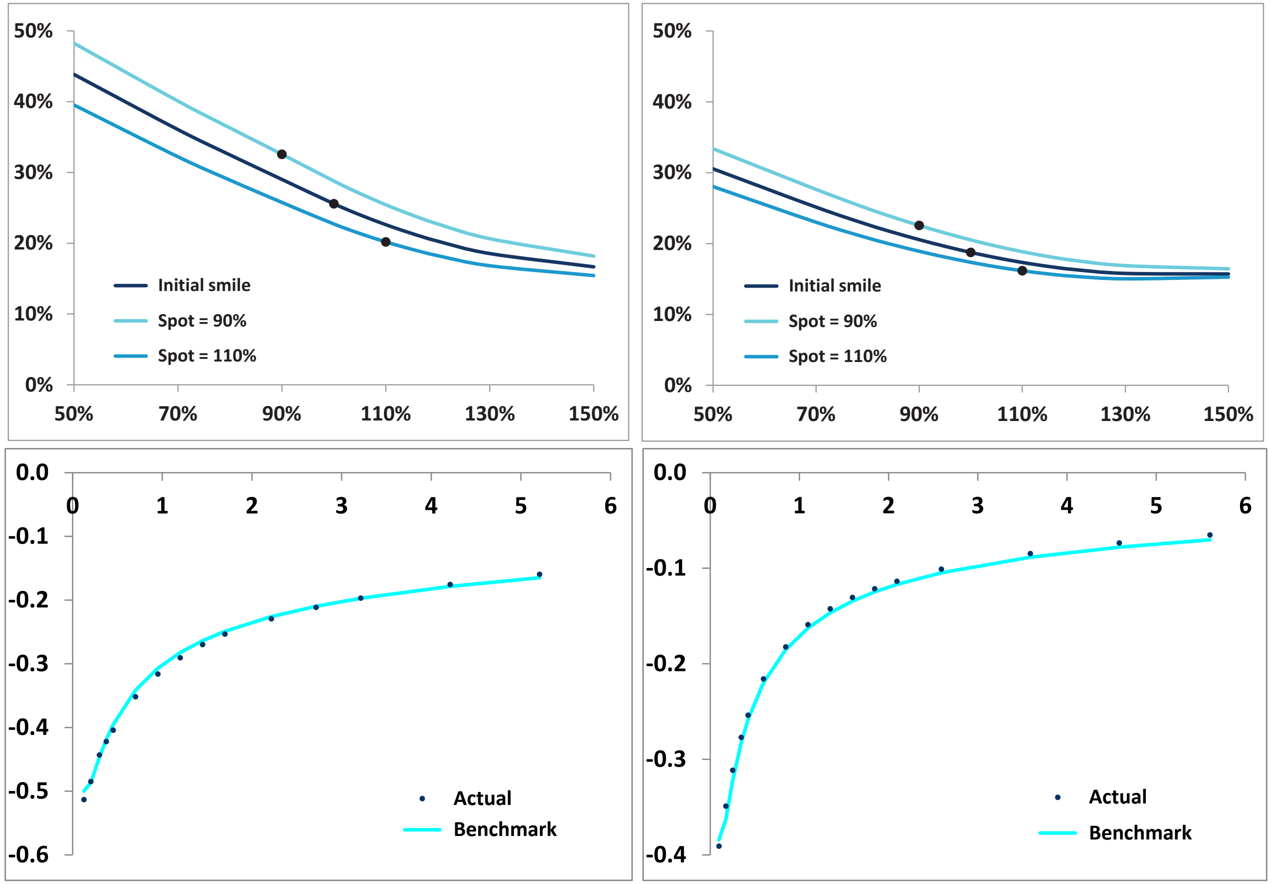


Figure: Top: smiles of the Eurostoxx50 index for a maturity $\simeq 1$ year observed on October 4, 2010 (left) and May 16, 2013 (right). Bottom: term structures of ATMF skew and power-law fits with $\gamma = 0.37$ (left), $\gamma = 0.52$ (right), as a function of T (years).

Dynamics in LV model – 5

- Real versus approximate SSR

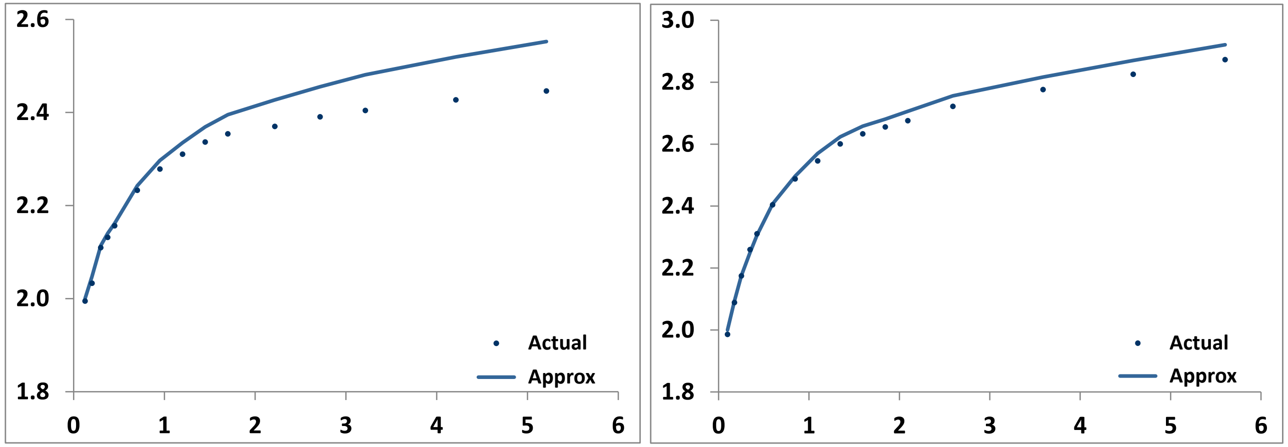
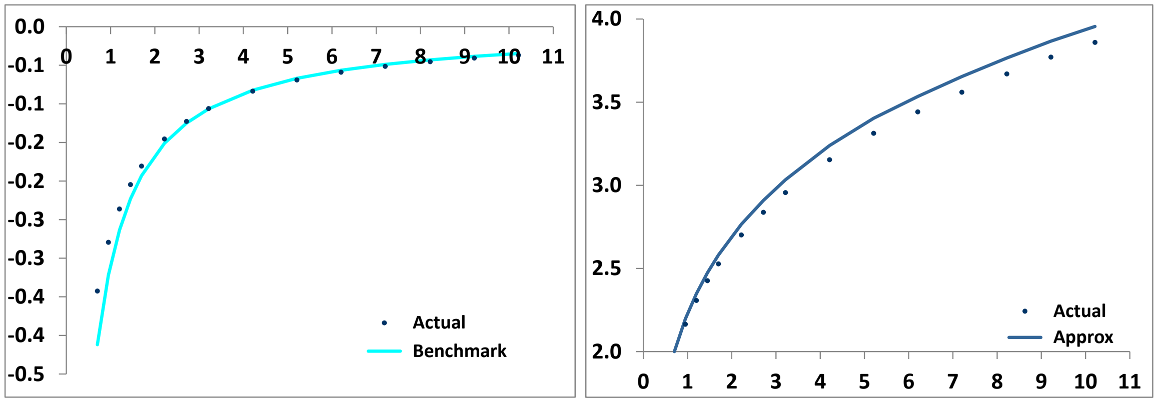


Figure: $\mathcal{R}_T$ as a function of T (years) computed: (a) in FD (actual), (b) using expression $\mathcal{R}_T = 1 + \frac{1}{T}\int_0^T \frac{\mathcal{S}_t}{\mathcal{S}_T}dt$ (approx).

- What about smile with $\mathcal{S}_T \propto \frac{1}{T}$? Approx fomula gives $\lim_{T\to\infty} \mathcal{R}_T = \infty$ (logarithmic divergence of $\mathcal{R}_T$):



- Approx slightly overestimates SSR.

Conclusion

- LV model is a genuine market model for underlying + vanilla options
- The only diffusive market model that possesses a 1-d Markov representation in terms of (t,S)
- Generates well-defined break-even levels for spot/vol and vol/vol covariances in the carry P&L.
- Daily recalibration of LV function – an ancillary object – is exactly how model should be used and deltas calculated.
  - Spot/vol break-even correlations $= -100\%$, vol/vol break-even correlations $= 100\%$.
  - Volatilities of implied volatilities given by: $\text{vol}(\widehat{\sigma}_{KT}) \; = \; \frac{1}{\Sigma_{KT}^{\text{LV}}} \frac{d\Sigma_{KT}^{\text{LV}}}{dS} S\sigma(t,S)$.
- Delta is well-defined: $\Delta^{\text{MM}} \; = \; \frac{d\mathcal{P}}{dS}\Big|_{O_{KT}}$. Delta of vanilla option irrelevant notion.
- When vega-hedging with (BS) delta-hedged vanilla options, sticky-strike delta should be used.
- Good approximate formulae for sizing up break-even vols of ATMF vols – or equivalently SSR:

$$\mathcal{R}_T \; = \; 1 + \frac{1}{T}\int_0^T \frac{\mathcal{S}_t}{\mathcal{S}_T} dt$$

$$\text{vol}(\widehat{\sigma}_{F_T T}) \; = \; \mathcal{R}_T \mathcal{S}_T \left(\frac{\widehat{\sigma}_{F_0 0}}{\widehat{\sigma}_{F_T T}} \right)$$

Local-stochastic volatility models – and non-models

Motivation

- In LV model, nothing to enter beside values of hedge instruments – zero parameter.
- Break-even covariances are set by prevailing smile. If smile is flat, implied vols of vols = 0.
- Can we regain some leverage on the model-implied dynamics of hedge instruments?

- Poor man's fix:
  - Pick your favourite stochastic volatility model.
  - Decorate SV instantaneous volatility with local volatility component.

- Is it a (usable) model?

- Provided answer is positive
  - What is the delta? What are the vegas?
  - What kind of model is it?

SV models

- Which SV model should we use?

- Unlike LV model, SV models have parameters that we can use to drive the dynamics of the $\widehat{\sigma}_{KT}$.

- First generation of SV models: based on instantaneous variance V_t, e.g. the Heston model:

$$\begin{cases} dS_t &= (r-q)S_t dt + \sqrt{V_t}\; S_t dW_t \\ dV_t &= -k(V_t - V^0)dt + \nu\sqrt{V_t} dZ_t \end{cases}$$

- Pbm: V_t not an asset – no way to generate $P\&L \propto (V_{t_2} - V_{t_1})$ ⇨ dynamics of $\widehat{\sigma}_{KT}$ needs to be checked a posteriori.

- Better to model dynamics of hedge instruments directly, for example forward variances ξ_t^T:

$$\xi_t^T = E_t\left[\left(\frac{dS_T}{S_T}\right)^2\right] = E_t[V_T]$$

  - Can be bought/sold by trading variance swaps (VS) – at no cost. VS volatility for maturity T at time t, $\widehat{\sigma}_T(t)$ given by:

$$\widehat{\sigma}_T^2(t) = \frac{1}{T-t}\int_t^T \xi_t^T dt$$

  - ξ_t^T is driftless: $d\xi_t^T = \bullet\, dW_t^T$

Forward variance models

- Need to specify a dynamics for the curve ξ_t^T such that:
  - Low-dimensional Markov representation
  - Able to generate flexible patterns for volatilities of VS volatilities $\widehat{\sigma}_T$. Typically:

$$\text{vol}(\widehat{\sigma}_T) \propto \frac{1}{T^\alpha}, \quad \alpha \in [0.3, 0.6]$$

- In practice using two Brownian motions with exponential weightings is sufficient:

$$\frac{d\xi_t^T}{\xi_t^T} = (2\nu)\mathcal{N}\left[(1-\theta)e^{-k_1(T-t)}dW_t^1 + \theta e^{-k_2(T-t)}dW_t^2\right]$$

with ν: volatility of a volatility with vanishing maturity and $\mathcal{N}$ normalization factor.

$$\xi_t^T = f^T(t, X_t^1, X_t^2)$$

with X_t^1, X_t^2 two OU processes – easily simulated exactly.

- Process for S_t is:

$$dS_t = (r-q)S_t dt + \sqrt{\xi_t^t} S_t dW_t^S$$

- Also able to generate decay of ATMF skew $\mathcal{S}_t \propto \frac{1}{T^\gamma}$ with γ typically $\approx \frac{1}{2}$. see papers Smile Dynamics II, III, IV.

Models used as examples in presentation

- Mixed Heston model

$$\begin{cases} dS_t &= (r-q)S_t dt + \boldsymbol{\sigma(t,S_t)}\sqrt{V_t}\ S_t dW_t \\ dV_t &= -k(V_t - V^0)dt + \nu\sqrt{V_t}dZ_t \end{cases}$$

- Mixed two-factor model

$$\begin{cases} dS_t &= (r-q)S_t dt + \boldsymbol{\sigma(t,S_t)}\sqrt{\zeta_t^t}\ S_t dW_t^S \\ \frac{d\zeta_t^T}{\zeta_t^T} &= 2\nu\ \mathcal{N}\left[((1-\theta)\,e^{-k_1(T-t)}dW_t^1\ +\ \theta e^{-k_2(T-t)}dW_t^2\right] \end{cases}$$

where $\alpha_\theta\ =\ 1/\sqrt{(1-\theta)^2+\theta^2+2\rho\theta\,(1-\theta)}$; ν vol of short vol.

- LV component $\sigma(t,S)$ calibrated on vanilla smile.

- Pricing function in mixed model:
  - in Heston model $P^{\mathrm{M}}\left(t,S,\sigma,\ V\right)$, V number.
  - in two-factor model $P^{\mathrm{M}}\left(t,S,\sigma,\ \zeta^u\right)$, ζ^u curve.

Usage of mixed models

- Choose model parameters & initial values of state variables:
  - In Heston: $\left(k, \sigma, \rho, V^0\right)$, V.
  - In two-factor model: $(k_1, k_2, \theta, \nu, \rho_{12}, \rho_{S1}, \rho_{S2})$, ζ.
- Calibrate local volatility function $\sigma(t, S)$ to market smile.
  - In 1-factor model like Heston: solve fwd PDE for density.
  - General technique: particle method of P. Henry-Labordère/J. Guyon (2009).
- Then Shift+F9 ⇨ produces a (real) number. Is it a price?
- What about deltas?
  - Typically, move spot, recalibrate local vol and reprice.
  - Is it right delta? What kind of carry P&L does this materialize?
- Let's assume this is a model. Can we have an approximate way of sizing up:
  - volatilities of implied vols
  - covariances of spot and implied vols – equivalently SSR?

Two pricing functionals

- $P^M(t,x)$: takes as inputs t, S, *LV function* $+$ state variables λ of SV model:

$$P^M\left(t, S, \sigma(,), \ \lambda\right)$$

  - In Heston: $\lambda = V$ – number
  - In two-factor model: $\lambda = \zeta^u$ – curve

- $P(t,\widehat{x})$ takes as inputs t, S, *implied vols* $+$ state variables of SV model:

$$P\left(t, S, \widehat{\sigma}_{KT}, \ \lambda\right)$$

- Could include in $x, \widehat{x}$ model parameters as well ($\equiv$ state variables with zero drift/vol).

- Will use $P(\ )$ – rather than $P^M(\ \ )$ – to do P&L accounting.
  - Could use prices rather than implied vols.

Carry P&L – $P(t, S, \widehat{\sigma}_{KT},\ \lambda)$

- In mixed model – for a set LV function – $\widehat{\sigma}_{KT}$ is a function of $t, S, \sigma(,) +$ state variables: $\widehat{x} \ = \ \widehat{x}(t, x)$:

$$P^{\mathrm{M}}(t, x) \ = \ P(t, \widehat{x}(t, x))$$

- Implied vols given by:

$$\widehat{\sigma}_{KT} \ \equiv \ \Sigma^{M}_{KT}(t, S, \sigma,\ \lambda)$$

P^{M}, P related through:

$$P^{\mathrm{M}}(t, S, \sigma,\ \lambda) \ = \ P\Big(t, S, \Sigma^{M}_{KT}(t, S, \sigma, \lambda),\ \lambda\Big)$$

- Pricing equation for P^{M} – with set LV function – zero rates:

$$\frac{dP^{\mathrm{M}}}{dt} + \left(\Sigma_k \mu_k \frac{d}{dx_k} + \frac{1}{2}\Sigma_{kl} a_{kl} \frac{d^2}{dx_k dx_l}\right) P^{\mathrm{M}} \ = \ 0$$

Carry P&L – 2

- Switch to variables $\widehat{x}$:

$$\frac{dP}{dt} + \left(\Sigma_i \widehat{\mu}_i \frac{d}{d\widehat{x}_i} + \frac{1}{2}\Sigma_{ij}\widehat{a}_{ij}\frac{d^2}{d\widehat{x}_i d\widehat{x}_j}\right)P = 0$$

  with:

$$\begin{cases} \widehat{\mu}_i = \dfrac{d\widehat{x}_i}{dt} + \Sigma_k \mu_k \dfrac{d\widehat{x}_i}{dx_k} + \dfrac{1}{2}\Sigma_{kl} a_{kl} \dfrac{d^2\widehat{x}_i}{dx_k dx_l} \\ \widehat{a}_{ij} = \Sigma_{kl} a_{kl} \dfrac{d\widehat{x}_i}{dx_k}\dfrac{d\widehat{x}_j}{dx_l} \end{cases}$$

- $\widehat{\mu}_i$ drift of $\widehat{x}_i$ and $\widehat{a}_{ij}$ covariance matrix of $\widehat{x}_i$ and $\widehat{x}_j$
  - as generated by mixed model *with fixed LV function*.
  - $\frac{d\widehat{x}_i}{dx_k}$ involve derivatives of functional $\Sigma^M_{KT}(t, S, \sigma, \lambda)$ with respect to t, S, λ.

- Now consider P&L of short option position – unhedged for now – zero rates:

$$P\&L = -P(t + \delta t,\ \widehat{x} + \delta\widehat{x}) + P(t, \widehat{x})$$

Carry P&L – 3

- Expand at order two in $\delta\widehat{x}$, one in δt.

$$P\&L = -\frac{dP}{dt}\delta t - \Sigma_i \frac{dP}{d\widehat{x}_i}\delta\widehat{x}_i - \frac{1}{2}\Sigma_{ij}\frac{d^2P}{d\widehat{x}_i d\widehat{x}_j}\delta\widehat{x}_i\delta\widehat{x}_j$$

$$= -\Sigma_i \frac{dP}{d\widehat{x}_i}\left(\delta\widehat{x}_i - \widehat{\mu}_i\delta t\right) - \frac{1}{2}\Sigma_{ij}\frac{d^2P}{d\widehat{x}_i d\widehat{x}_j}\left(\delta\widehat{x}_i\delta\widehat{x}_j - \widehat{a}_{ij}\delta t\right)$$

- Among components of $\widehat{x}$:
  O_i market observables: $S, \widehat{\sigma}_{KT}$.
  λ_k = state variables of SV model

- Rewrite P&L:

$$P\&L = -\Sigma_i \frac{dP}{dO_i}\left(\delta O_i - \widehat{\mu}_i\delta t\right) - \frac{1}{2}\Sigma_{ij}\frac{d^2P}{dO_i dO_j}\left(\delta O_i\delta O_j - \widehat{a}_{ij}\delta t\right)$$

$$-\Sigma_k \frac{dP}{d\lambda_k}\left(\delta\lambda_k - \widehat{\mu}_k\delta t\right)$$

$$-\frac{1}{2}\Sigma_{kl}\frac{d^2P}{d\lambda_k d\lambda_l}\left(\delta\lambda_k\delta\lambda_l - \widehat{a}_{kl}\delta t\right) - \Sigma_{ik}\frac{d^2P}{dO_i d\lambda_k}\left(\delta O_i\delta\lambda_k - \widehat{a}_{ik}\delta t\right)$$

P&L hedged portfolio

- Portfolio: option + hedges that offset sensitivities $\frac{dP}{dO_i}$: $P_{\mathrm{H}} = P + \Sigma_i \alpha_i f_i(t, S, O_i)$
- P&L equation also holds for hedge instruments ⇨ canceling δO_i term cancels $\widehat{\mu}_i \delta t$ contribution. P&L of hedged position:

$$
\begin{aligned}
P\&L_{\mathrm{H}} \;=\; & -\frac{1}{2}\Sigma_{ij}\frac{d^2 P_{\mathrm{H}}}{dO_i dO_j}\left(\delta O_i \delta O_j - \widehat{a}_{ij}\delta t\right) \\
& - \Sigma_k \frac{dP_{\mathrm{H}}}{d\lambda_k}\left(\delta\lambda_k - \widehat{\mu}_k \delta t\right) \\
& - \frac{1}{2}\Sigma_{kl}\frac{d^2 P_{\mathrm{H}}}{d\lambda_k d\lambda_l}\left(\delta\lambda_k \delta\lambda_l - \widehat{a}_{kl}\delta t\right) - \Sigma_{ik}\frac{d^2 P_{\mathrm{H}}}{dO_i d\lambda_k}\left(\delta O_i \delta\lambda_k - \widehat{a}_{ik}\delta t\right)
\end{aligned}
$$

- 1st piece OK: thetas matching gammas on market instruments.
    - $\widehat{a}_{ij}$ positive covariance matrix: $\widehat{a}_{ij} \;=\; \Sigma_{kl} a_{kl} \frac{d\widehat{x}_i}{dx_k}\frac{d\widehat{x}_j}{dx_l}$
- 2nd / 3d pieces no good. P&L leakage from variation (or not) of SV state variables.
- By construction value of hedges indpdt on λ_k: $\frac{df_i}{d\lambda_k} = 0$, so:

$$
\frac{dP_{\mathrm{H}}}{d\lambda_k} = \frac{dP}{d\lambda_k}, \qquad \frac{d^2 P_{\mathrm{H}}}{d\lambda_k d\lambda_l} = \frac{d^2 P}{d\lambda_k d\lambda_l}, \qquad \frac{d^2 P_{\mathrm{H}}}{dO_i d\lambda_k} = \frac{d^2 P}{dO_i d\lambda_k}
$$

P&L hedged portfolio – 2

- $\delta\lambda_k$ are not market values – are in our control. For example, take $\delta\lambda_k = \widehat{\mu}_k \delta t$.
- Still leaves us with 3d piece in P&L:

$$P\&L_{\mathrm{H}}^{\mathrm{leak}} = -\frac{1}{2}\Sigma_{kl}\frac{d^2P}{d\lambda_k d\lambda_l}\left(\delta\lambda_k\delta\lambda_l - \widehat{a}_{kl}\delta t\right) - \Sigma_{ik}\frac{d^2P}{dO_i d\lambda_k}\left(\delta O_i\delta\lambda_k - \widehat{a}_{ik}\delta t\right)$$

- Is there a solution to P&L leakage?
- YES – need condition on $P(t, O,\ \lambda)$:

$$\left.\frac{dP}{d\lambda_k}\right|_{S,\widehat{\sigma}_{KT}} = 0,\ \forall k$$

- Pricing functional $P(t, S, \widehat{\sigma}_{KT},\ \lambda)$ must have zero sensitivity to SV state variables.

Conclusion: admissible (or gauge-invariant) models

- Criterion for models that can be used in trading: $P\left(t, S, \widehat{\sigma}_{KT},\ \lambda\right)$:

$$\left.\frac{dP}{d\lambda_k}\right|_{S,\widehat{\sigma}_{KT}} \;=\; 0$$

- $P\&L_{\mathrm{H}}$ of delta-hedged/vega-hedged position then has typical form of market models:

$$P\&L_{\mathrm{H}} \;=\; -\,\frac{1}{2}\Sigma_{ij}\,\frac{d^2 P_{\mathrm{H}}}{dO_i dO_j}\left(\delta O_i \delta O_j - \widehat{a}_{ij}\delta t\right)$$

  Break-even covariance levels are given by covariances in model *with fixed LV function*: $\widehat{a}_{ij} \;=\; \Sigma_{kl}\, a_{kl}\, \frac{d\widehat{x}_i}{dx_k}\frac{d\widehat{x}_j}{dx_l}$.

- Pbm: condition $\left.\frac{dP}{d\lambda}\right|_{S,\widehat{\sigma}_{KT}} = 0$ usually *not* satisfied.
  - Ex: not satisfied in local/stoch vol model built on Heston model: $\frac{d}{dV}P\left(t, S, \widehat{\sigma}_{KT},\ V\right) \neq 0$ ⇨ P&L leakage
  - Not usable in trading.

- Do admissible models exist at all?
  - YES.

Admissible models – 2

- Consider mixed two-factor model. Pricing function $P(t, S, \widehat{\sigma}_{KT},\ \zeta^u)$.
- Model equivalently written as:

$$\begin{cases} dS_t &= (r-q)S_t dt + \sqrt{\zeta^t}\sqrt{f(t, X_t^1, X_t^2)}\ \sigma(t, S_t)\ S_t dW_t^S \\ dX_t^1 &= -k_1 X_t^1 dt + dW_t^1 \\ dX_t^2 &= -k_2 X_t^2 dt + dW_t^2 \end{cases}$$

with $X_0^1 = 0, X_0^2 = 0$ and:

$$f(t, x_1, x_2) = e^{2\nu\alpha_\theta\left[(1-\theta)x_1 + \theta x_2\right] - \frac{(2\nu\alpha_\theta)^2}{2}\chi(t)}$$

$$\chi(t) = (1-\theta)^2 \frac{1-e^{-2k_1 t}}{2k_1} + \theta^2 \frac{1-e^{-2k_2 t}}{2k_2} + 2\rho\theta(1-\theta)\frac{1-e^{-(k_1+k_2)t}}{k_1+k_2}$$

- Pick arbitrary φ^u, do following transformation:

$$\zeta^u \rightarrow \varphi^u \zeta^u$$

$$\sigma(u, S) \rightarrow \sqrt{\frac{1}{\varphi^u}}\sigma(u, S)$$

- SDEs for S_t, X_t^1, X_t^2 unchanged: $\frac{\delta P}{\delta \zeta^u} = 0$ ⇨ mixed two-factor model admissible.

Admissible models – 3

- Other admissible models:
  - lognormal model for V_t (SABR)
  - smiled version of two-factor model (see SD III)
- Significance of condition $\left.\frac{dP}{d\lambda}\right|_{S,\widehat{\sigma}_{KT}} = 0$
  - $\left.\frac{dP}{d\lambda}\right|_{S,\widehat{\sigma}_{KT}} \neq 0$: price depends on more state variables than hedge instruments. Ex. with Heston model: $P\left(t, S, \widehat{\sigma}_{KT},\ V\right)$.
  - State variables λ are stochastic ⇨ model allocates thetas proportional to $\frac{d^2P}{d\lambda^2}, \frac{d^2P}{d\lambda dO}$ ⇨ P&L leakage, even if $\delta\lambda = 0$.
  - Does not happen with model parameters V^0, k, ν, ρ – do not generate P&L leakage.
    - Model allocates no theta to gammas on model params.
  - Like making P a function of a non-financial state variable – e.g. temperature.
- In admissible models, SV degrees of freedom *do* impact dynamics of assets, yet do not require extra hedges.

Now know which models are usable – what's left to do?

- Size up break-even covariance levels for $S/\widehat{\sigma}_{KT}$, $\widehat{\sigma}_{KT}/\widehat{\sigma}_{K'T'}$.
  - Like them, use model; don't like them, don't use model.
  - In practice, look at dynamics of implied vols with floating strikes – fixed moneyness, rather than fixed strikes.

- Approximate formulae for vols of vols and spot/vol covariances – for ATMF vols?

- Consider in particular SSR:

$$\mathcal{R}_T = \frac{1}{\mathcal{S}_T} \frac{\langle d\widehat{\sigma}_T d\ln S\rangle}{\langle (d\ln S)^2\rangle}$$

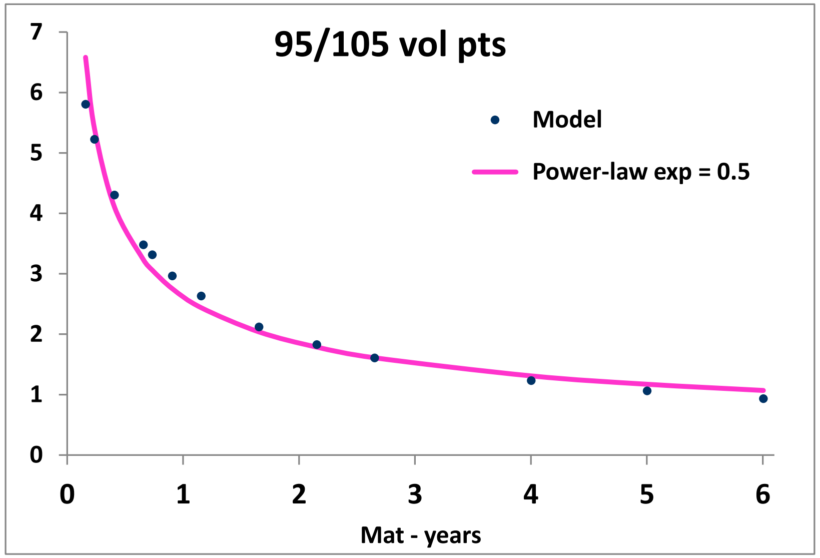
- Expand at order one in vol of vol ν and local vol function.

Example

- Pick as mkt smile smile generated by two-factor model. Parameters typical of Eurostoxx50 smile. VS vols flat at 20%.
  - So that full SV situation attainable.
- Parameters so that vol$(\widehat{\sigma}_T) \propto \frac{1}{T^{0.6}}$.
- ρ_{SX^1}, ρ_{SX^2} (calibrated on actual smile) so that $\mathcal{S}_T \simeq \frac{1}{T^{0.5}}$.

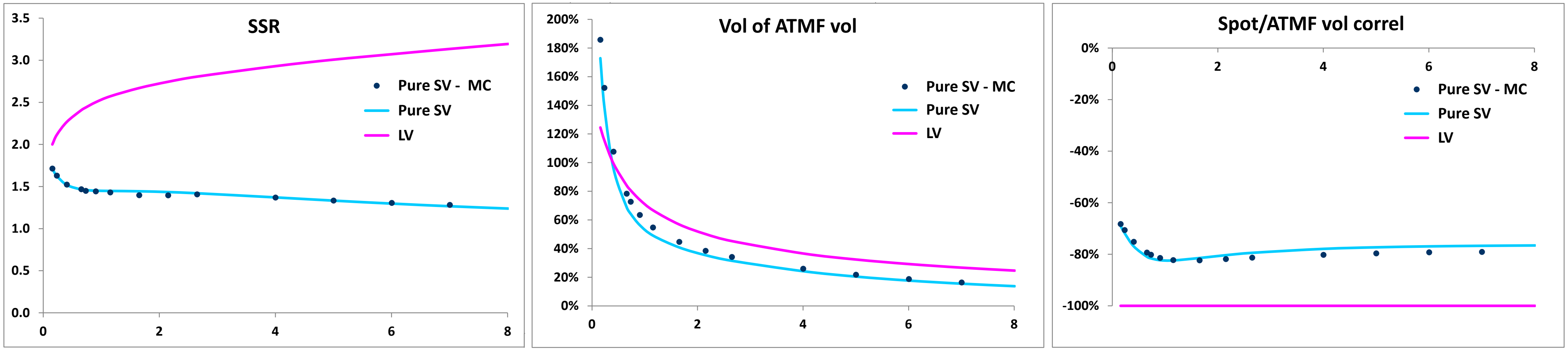
Model params	
nu	310.0%
theta	13.9%
k1	8.59
k2	0.47
rho XY	0%

rho_SX	-54.0%
rho_SY	-62.3%

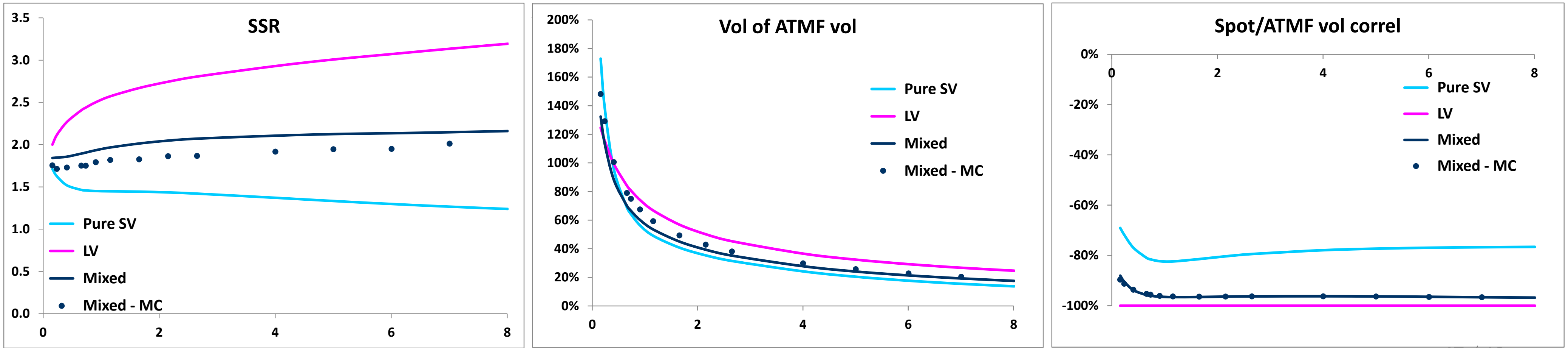


Example – 2

- **Test 1**: use same parameters for underlying SV model – local vol flat = 1.
  - MC: computed numerically – other curves: order-1 formulae
  - Everything as function of maturity (years).



- **Test 2**: Now halve vol of vol of underlying SV model



Conclusion

- Characterization of local/stoch vol models that can be used for trading. Pricing function $P(t, S, \widehat{\sigma}_{KT},\ \lambda)$ has to be such that:

$$\left.\frac{dP}{d\lambda}\right|_{S,\widehat{\sigma}_{KT}} = 0$$

  Models not obeying this condition ➩ P&L leakage.

- Models obeying condition are genuine market models: thetas matching asset/asset cross-gammas with positive break-even covariance matrix.

- Delta and vega given simply by $\left.\frac{dP}{dS}\right|_{\widehat{\sigma}_{KT}}$ and $\left.\frac{dP}{d\widehat{\sigma}_{KT}}\right|_{S}$ – the LV function is recalibrated.

- Good approximate expressions for break-even covariances for ATMF vols & spot.